Motion with Constant Acceleration

Key Ideas

- For some interesting physical situations, assuming the acceleration of an object is constant for a given time interval is a reasonable approximation. This is the **Constant Acceleration Model**.
- In the Constant Acceleration Model, there are well-defined relations between time, displacement, velocity, and acceleration for an object.
- Physical problems where the acceleration is approximately constant for successive time intervals and changes quickly between intervals can be solved by applying the Constant Acceleration Model during each time interval.

Learning Objectives

By the end of this section, you should:

- understand the relationships between the kinematic quantities, displacement, velocity, acceleration, and time for the Constant Acceleration Model,
- use motion graphs of position, velocity, and/or acceleration vs time to help solve problems where acceleration is constant,
- use a structured approach to conceptualize, set up, solve, and check constant acceleration problems, and
- solve kinematics problems algebraically, identifying known and unknown quantities, before calculating numerical results.
- solve problems with multiple time intervals with different accelerations by connecting the final properties for one time interval with the initial properties of the next time interval.

In the previous two sections, we defined the relationships between position and velocity (in Velocity and Speed) and the relationships between velocity and acceleration (in Motion Along a Straight Line - Acceleration). There are, of course, connections among all four kinematic properties (properties related to motion), position, velocity, acceleration, and time. For example, you might guess that the greater the acceleration of, say, a car moving away from a stop sign, the greater the car's displacement in a given time. These all come from the definitions for velocity as the rate of change with time of position, or the derivative of $\vec{x}(t)$, and for acceleration as the rate of change with time of the velocity, or the derivative of $\vec{v}(t)$.

In this section, we consider a specific physical model, the **Constant Acceleration Model**, where we'll derive algebraic equations for these kinematic relationships. Remember that, in physics, a "model" is a set of one or more approximations that allow us to solve specific types of problems. When using a model, we must always be aware of the approximations made. In this case, the analytic results of the model ***only apply during a time interval where the acceleration doesn't change, to a good approximation***. This seems pretty obvious, considering the "Constant Acceleration" name for the model.

Assuming acceleration to be constant is a big approximation for most motion and generally not applicable, although it is reasonably accurate in some cases. It can be used to find approximate results for changes in position and velocity by assuming a constant acceleration equal to the average acceleration. This is most useful for time intervals where the acceleration does not change much. Finally, this model is useful to describe motion that can be considered in separate time intervals, each of which has an approximately constant acceleration.

We will start by setting up the type of problem being solved and defining the notation to be used. Then, starting from the definitions of velocity and acceleration, we'll derive the constant acceleration **Kinematic Equations for Constant Acceleration**. These will then be applied to physical problems: (i) a single object in motion with constant acceleration, (ii) a single object in motion where the acceleration has different, constant values during successive time intervals, and (iii) the motion of two objects moving relative to each other, sometimes referred to as **pursuit** or **chase** problems.

Notation

In this chapter we are considering motion in only one dimension. We will use a coordinate system with a defined origin and positive x-direction (although this direction might just as well be called "y" or "z"). The quantities we'll be using are the x-components of the position, velocity, and acceleration vectors, $x(t),\ v_x(t),\ a_x(t) = a_x$, for a constant acceleration. As x-components of the position, velocity, and acceleration vectors, these can be positive, negative, or zero, the sign indicating the direction. This will be generalized to motion in more than one dimension in the next chapter.

Defining the quantities and notation we'll be using:

- Times: Initial time for the problem, t_i; Final time for the problem, t_f; Times during the motion, $t \text{ with } t_i \le t \le t_f$; Time interval for the problem, $\Delta t = t_f - t_i$
- Acceleration: $a_x = \text{constant}$
- Positions: Position function $x(t)$; Initial position, $x_i = x(t_i)$; Final position, $x_f = x(t_f)$; Total displacement, $\Delta x = x_f - x_i$
- Velocities: Velocity function $v_x(t)$; Initial velocity, $v_{x,i} = v_x(t_i)$; Final velocity, $v_{x,f} = v_x(t_f)$; Change in velocity, $\Delta v = v_{x,f} - v_{x,i}$

Note: In the derivations below, the x-component subscripts will be dropped to make the notation somewhat easier to follow. The following replacements will be made:

$$v_x(t) \Rightarrow v(t),\ \ v_{x,i} \Rightarrow v_i,\ \ v_{x,f} \Rightarrow v_f,\ \ a_x \Rightarrow a$$

This can be done for one-dimensional motion where these vectors only have x-components. For motion in two- or three-dimensions, it is important to always distinguish the different directions, meaning the different components of these vectors. In these cases, using correct subscripts is very important.

Next we'll determine equations that give the kinematic relations between acceleration, velocity, position, and time by taking an algebraic and graphical approach. There are other ways to determine these relationships. For example, in the next section, we will use integral calculus.

Relating Velocity, Acceleration, and Time

Figure 3.19 is a graph of $v_x(t)$ for an object moving with a constant acceleration. Because the acceleration is the slope of $v_x(t)$, and constant, the velocity graph of $v(t)$ is a straight line.

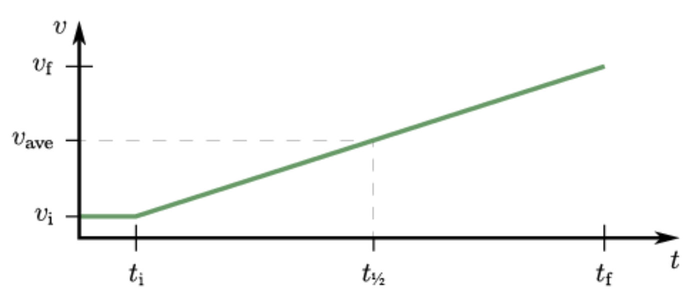

Figure 3.19 Velocity vs. time graph for constant acceleration from a time t_i to t_f. The average velocity for the time interval shown is the

velocity at the time in the middle of the interval.

For constant acceleration, the average acceleration for this time interval is equal to the acceleration at any time. Using the definition of the average acceleration, where we'll use $a = a_{ave}$, it is possible to solve for the final velocity in terms of the initial velocity, acceleration, and time.

$$\begin{aligned} a &= \frac{v_f - v_i}{t_f - t_i} \\ v_f &= a\left(t_f - t_i\right) + v_i \end{aligned} \qquad (3.20)$$

This result means that we can know the velocity at the end of the time interval if we know the initial velocity and the acceleration. We can predict the future!

In fact, there is more that can be determined from Equation 3.20. Because $v(t)$ is a straight line, with a constant slope, for all times in the time interval shown, the equation for the average acceleration, which is constant, holds for any time $t_i < t \leq t_f$, not just t_f, giving an equation for $v(t)$ at any time.

Velocity from Acceleration and Time

For constant acceleration, a, in a time interval $t_i \leq t \leq t_f$, the velocity is

$$v(t) = v(t_i) + a(t - t_i) \qquad (3.21)$$

Considering the full time interval, $\Delta t = t_f - t_i$, the final velocity is

$$\begin{aligned} v_f &= v_i + a\Delta t \\ \Delta v &= v_f - v_i = a\Delta t \end{aligned} \qquad (3.22)$$

This indicates, as expected, that the change in the magnitude of the velocity increases as either the magnitude of the acceleration increases or the time interval increases, making physical sense. If the acceleration is zero, the velocity doesn't change.

Relating Position, Velocity, and Time

The definition of average velocity can be used to relate position to velocity and time, similar to what was done above with average acceleration. We must be careful, however, in determining the average velocity. The average being used is the average in time. Figure 3.20 shows that the average velocity during the time interval of interest, v_{ave}, is the mid-point between v_i and v_f. The velocity is less than the average for the same amount of time as it is greater than the average. This is generally true ***only*** for constant acceleration where $v(t)$ is a straight line with constant slope. Figure 3.20 gives an example of a velocity with a non-constant acceleration (one that increases with elapsed time). It is clear in this case that the average velocity is closer to v_i than v_f, not the average of the two, because the velocity is lower for a longer period of time.

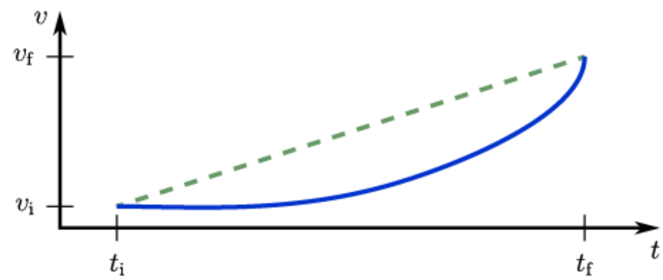

Figure 3.20 Velocity vs time graph for an object moving with a non-constant acceleration (lower, blue line). The average velocity is close to the initial velocity, v_i, because of the longer period of time when the velocity of the object is close to the initial velocity. Motion with a constant acceleration, with the same initial and final velocities, is shown as the dashed, green line for comparison.

Note: In Finding Velocity and Displacement from Acceleration, a more precise, mathematical definition of average velocity will be given using integrals.

Setting the average of v_i and v_f equal to the definition of average velocity, the displacement divided by the time interval, gives an equation for the final position of the object:

$$\begin{aligned} v_{ave} &= \frac{1}{2}(v_f + v_i) = \frac{x_f - x_i}{t_f - t_i} \\ x_f &= \frac{1}{2}(v_f + v_i) \cdot (t_f - t_i) + x_i \end{aligned} \tag{3.23}$$

This can also be written as: $\Delta x = \frac{1}{2}(v_f + v_i)\,\Delta t$

Just as we considered previously, the definition of the "final" time is arbitrary, as long as the acceleration is constant. That means the results of Equation 3.23 will apply for any time in the time interval considered, $t_i < t \leq t_f$.